**Nonlocal Transformation Optics – Supplementary Material**

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Note: Newly introduced equations are labeled with the prefix "S"; all others (as well as all references and figures) pertain to the actual paper.

## 1. Determination of the coordinate-transformation parameters

By substituting (11) in (5) [with (10a)], we obtain a biquadratic equation in the unknown $k_{tz}$, whose real, positive solutions of interest are given by:

$$k_{tz1,2} = \sqrt{\frac{-b_0 \pm \sqrt{b_0^2 + 4b_2 k_0^2 \left(1 - a_0 \sin^2\theta_i - a_2 k_0^2 \sin^4\theta_i\right)}}{2b_2}}, \tag{S1}$$

where the square-root arguments are assumed positive. Together with the known (conserved) tangential components [cf. (10a)], (S1) relates the transmitted wavevectors $\boldsymbol{k}_{t1}$ and $\boldsymbol{k}_{t2}$ to the incident wavevector and the coordinate transformation in (11).
By enforcing (S1) in (10b), and recalling (9) and (11), we obtain the following system of two algebraic equations

$$\frac{k_0 \sin\theta_i \left(a_0 + 2a_2 k_0 \sin\theta_i\right)}{k_{tz1,2}\left(b_0 + 2b_2 k_{tz1,2}^2\right)} = \tan\theta_{t1,2}, \tag{S2}$$

which can be solved analytically in closed form, thereby allowing to express two coefficients (say $b_0$ and $b_2$) as a function of the remaining parameters,

$$\begin{cases} b_0 = \dfrac{\sin^2\theta_i \left(a_0 + 2a_2 k_0^2 \sin^2\theta_i\right)^2 \left(\tan^2\theta_{t1} + \tan^2\theta_{t2}\right)}{\left[1 - \sin^2\theta_i \left(a_0 + a_2 k_0^2 \sin^2\theta_i\right)\right]\left(\tan^2\theta_{t1} - \tan^2\theta_{t2}\right)^2}, \\ b_2 = \dfrac{-\sin^4\theta_i \left(a_0 + 2a_2 k_0^2 \sin^2\theta_i\right)^4 \tan^2\theta_{t1} \tan^2\theta_{t2}}{k_0^2 \left[1 - \sin^2\theta_i \left(a_0 + a_2 k_0^2 \sin^2\theta_i\right)\right]^3 \left(\tan^2\theta_{t1} - \tan^2\theta_{t2}\right)^4}. \end{cases} \tag{S3}$$

Equations (S3) define a family of (infinite) coordinate transformations which satisfy the assigned kinematical characteristics of the two transmitted waves, for the given incidence conditions.

## 2. Synthesis of the PC approximant

The Bloch-type exact dispersion law for the 1-D multilayered PC of interest is given by:

$$\cos\left[k_x\left(d_a + d_b\right)\right] = \cos\left(k_{xa} d_a\right)\cos\left(k_{xb} d_b\right) - \frac{1}{2}\left(\frac{k_{xa}\varepsilon_b}{k_{xb}\varepsilon_a} + \frac{k_{xb}\varepsilon_a}{k_{xa}\varepsilon_b}\right)\sin\left(k_{xa} d_a\right)\sin\left(k_{xb} d_b\right), \tag{S4}$$

with $k_{xa,b} = \sqrt{k_{a,b}^2 - k_z^2}$ and $k_{a,b} = k_0\sqrt{\varepsilon_{a,b}}$.

In [9], a nonlocal homogenized model was developed in terms of a uniaxial medium whose dispersion law would match the exact dispersion law in (S4) up to the fourth order in $d_{a,b}$, with associated constitutive parameters similar to (12). In our case, we verified that, in view of the generally small dynamical ranges involved [cf. Fig. 2(c)], such approach yields a satisfactorily accurate modeling of the $k_x$ dependence in $\tilde{\varepsilon}_{zz}$. However, in order to accurately capture the $k_z$ dependence in $\tilde{\varepsilon}_{xx}$ over the generally much larger dynamical ranges involved [cf. Fig. 2(b)], we found that a higher-order model is needed. Accordingly, we developed a modified nonlocal homogenized model:

$$\tilde{\varepsilon}_{xx}^{(\mathrm{hom})}(k_z) = \frac{1}{\beta_0 + \beta_2 k_z^2 + \beta_3 k_z^4 + \beta_6 k_z^6}, \quad \tilde{\varepsilon}_{zz}^{(\mathrm{hom})}(k_x) = \frac{1}{\alpha_0 + \alpha_2 k_x^2 + \alpha_4 k_x^4 + \alpha_6 k_x^6}, \tag{S5}$$

whose coefficients are obtained by matching the exact dispersion law in (S4) up to the tenth-order in $d_{a,b}$. The analytical expressions of the coefficients $\alpha_n$ and $\beta_n$ (in terms of $k_0$ and the PC parameters $\varepsilon_a$, $\varepsilon_b$, $d_a$, and $d_b$) can be straightforwardly obtained via symbolic manipulation softwares (e.g., Mathematica, www.wolfram.com), and are not reported here for brevity.

The final step of the procedure consists of determining the parameters of the PC approximant by matching the nonlocal homogenized model in (S5) with the TO-based blueprints in (12). We verified that, as expected from the above observations, the coefficients $\alpha_4$ and $\alpha_6$ in (S5) are actually negligible over the parametric ranges of interest, so that the nonlocal-homogenized model and TO-based blueprint for $\tilde{\varepsilon}_{zz}$ are functionally identical, and the corresponding matching can be enforced analytically via:

$$\alpha_0 = a_0, \quad \alpha_2 = a_2. \tag{S6}$$

For $\tilde{\varepsilon}_{xx}$, we were instead led to use a numerical strategy relying on a downhill-simples-based minimization of the root-mean-square mismatch within ($\pm 10\%$) neighborhoods of the prescribed transmitted wavenumbers $k_{tz1}$ and $k_{tz2}$, with the constraint $(d_a + d_b) < \lambda_0/2$ so as to avoid propagation of higher-order Bragg modes. In this framework, we recall that the TO-based blueprints in (12), actually represent a family of *infinite* transformation media [with $a_0$ and $a_2$ that are in principle free, and $b_0$ and $b_2$ constrained by (S3)]. Such flexibility allows for a more effective parameter matching between the PC approximant and the TO-based blueprints. Overall, based on a large body of simulations, the proposed procedure was found to yield errors $\lesssim 10°$ between the prescribed and actual transmitted-wave directions.

### 3. FDTD simulations

The field map in Fig. 3 was obtained via an FDTD [29] simulation of a PC slab consisting of 890 unit cells [see the inset in Fig. 2(b)], with $\varepsilon_a = 2.752$, $d_a = 0.0668\lambda_0$, $\varepsilon_b = -2.082$, and $d_b = 0.0332\lambda_0$, with an overall size of $89\lambda_0$ (along *x*) × $15\lambda_0$ (along *z*). The PC slab was illuminated by a collimated Gaussian beam with waist of size $w_0 = 15\lambda_0$, impinging from a vacuum region vacuum ($z<0$) with an angle $\theta_i = 40°$ with respect to the $z$-axis. A uniform spatial discretization of step $\Delta_x = \Delta_z = \lambda_0/120$ was assumed, with time sampling $\Delta_t = \Delta_x\Big/\left(5\sqrt{2}c_0\right)$ (i.e., five-time smaller than the Courant stability limit). The computational domain was terminated using second-order Mur-type absorbing boundary conditions. At this stage, material losses were neglected in order to better highlight the wave-splitting phenomenon of interest. For the negative-permittivity layers, a plasma-type model

$$\varepsilon_b\left(\omega\right) = 1 - \frac{\omega_p^2}{\omega^2}, \tag{S7}$$

was used (with $\omega_p$=1.756$\omega$), and implemented via the auxiliary differential equation (ADE) method [29].