Efficient Numerical Analysis of Arbitrary Single-Mode Optical Fibers Using Padé Approximants

Vincenzo Fiumara, Vincenzo Galdi, *Member, IEEE,* Vincenzo Pierro, and Innocenzo M. Pinto

***Abstract*— We propose a new, fast, and accurate numerical technique for analyzing single-mode optical fibers with arbitrary (transverse) refractive index profile. The method is based upon a Padé (rational) approximation of the spectral domain Green's function of the fiber, obtained by solving a hierarchy of *static* problems. The sought eigenfrequency and modal field are accordingly estimated by computing, respectively, the dominant pole and the related residual of the rational approximant. Numerical simulations and comparison with known analytical results indicate that the proposed method is highly accurate, reliable, and computationally affordable.**

***Index Terms*—Optical fibers, Padé approximants.**

I. Introduction

THE ever-increasing interest in numerical modeling of optical fibers is witnessed by the huge amount of recent technical papers devoted to this subject [1]–[9]. Most of the fibers currently manufactured for long-distance communication systems are *weakly-guiding* and operate in the *single-mode* range. In this connection, the design of the profile index function is usually obtained by trial and error analysis so as to optimize the dispersion characteristics and matching them to the spectral features of specific sources. A practical implementation of this procedure requires a fast and reliable numerical tool for solving the scalar wave equation. As a matter of fact, analytical solutions are available only for few profiles; standard analytical approximation techniques (e.g., gaussian [10], [11]) cannot be applied for arbitrary profiles and are known to fail in the low-frequency limit, while available numerical techniques (including matrix approaches [3], polynomial expansions [4], [7], Galerkin methods [1], [5], [8], finite elements [9], and path-integrals [6]) become inefficient in terms of memory allocation and/or computing times if highly accurate results are required.

In this letter we present a novel approach, based upon a Padé approximation of the spectral domain Green's function, which is shown to provide a very accurate approximation for the dispersion law and the field distribution of the fundamental mode over the useful spectral range, for any (transverse) index distribution, with minimum storage requirements and CPU times.

Manuscript received May 7, 1998.

The authors are with the Dipartimento di Ingegneria dell'Informazione e Ingegneria Elettrica, University of Salerno, I-84084 Fisciano (SA), Italy.

Publisher Item Identifier S 1051-8207(98)07956-2.

II. The Method

A longitudinally invariant graded-index circular fiber with uniform cladding is considered. An $\exp(-j\omega t)$ time-harmonic dependence is implicitly assumed throughout the remainder of the paper. The refractive index profile is expressed as [10]

$$n^2(r) = n_{co}^2[1-2\Delta f(r)], \quad f(r) = \begin{cases} 0, & \text{for } r=0 \\ 1, & \text{for } r \geq \rho \end{cases} \tag{1}$$

where r is the radial coordinate and $\Delta = (1 - n_{cl}^2/n_{co}^2)/2$ is the so-called *profile height parameter,* n_{co}, n_{cl} being the maximum core index and the clad index, respectively.

It is expedient to introduce some normalized parameters:

$$V = k\rho(n_{co}^2 - n_{cl}^2)^{1/2}, \quad \text{normalized frequency} \tag{2}$$

$$U = \rho(k^2 n_{co}^2 - \beta^2)^{1/2}, \quad \text{core modal parameter} \tag{3}$$

$$W = \sqrt{V^2 - U^2}, \quad \text{cladding modal parameter} \tag{4}$$

where k is the free-space wavenumber and β is the propagation constant.

In the *weakly-guiding* regime ($n_{co} \approx n_{cl}$, i.e., $\Delta \ll 1$), the radial behavior of the HE_{1n} modes is ruled by the following Sturm–Liouville equation [10]:

$$\left[\frac{1}{R}\frac{d}{dR}\left(R\frac{d}{dR}\right) - W^2 + V^2 h(R)\right]\phi(R) = 0 \tag{5}$$

where $R = r/\rho$ and $h(R) = 1 - f(R)$.[1]

For any fixed positive value of W^2 (guided modes) the above problem admits a discrete infinity of eigenvalues $\{V_n^2\}$ and related eigenfunctions $\{\phi_n\}, n = 1, 2, 3, \cdots$, which form an orthonormal basis for $L^2(\mathcal{R}^+)$, i.e., [12]

$$\int_0^1 R\phi_n(R)\phi_m(R)h(R)\, dR = \delta_{nm} \tag{6}$$

δ_{nm} being the Kronecker symbol. Our aim is to find out the dispersion relation (usually represented as a U versus V diagram) and the fundamental mode (HE_{11}). To this end, let

[1] Extension to the more general vector (or coupled-scalar) problem, though possible in principle, entails considerable formal complications. In this letter we limit ourselves to the simpler (though highly meaningful) case of *weakly guiding* fibers to highlight the physical content of the method.

us consider the spectral domain Green's function (ring source) [12]:

$$\left[\frac{1}{R}\,\frac{d}{dR}\left(R\,\frac{d}{dR}\right) - W^2 + V^2 h(R)\right] G_V(R, R') = \frac{1}{R}\,\delta(R - R') \qquad (7)$$

which admits an expression in terms of the aforementioned eigenvalues and eigenfunctions, viz., [12]

$$G_V(R, R') = \sum_{n=1}^{\infty} \frac{\phi_n(R)\phi_n(R')}{V^2 - V_n^2}. \qquad (8)$$

Since G_V is *meromorphic*, Mittag-Leffler's theorem [13] suggests that it could be well approximated by a rational function of the (square of the) frequency, viz.,

$$G_V(R, R') \approx \frac{\sum_{i=0}^{P} a_i(R, R') V^{2i}}{1 + \sum_{i=1}^{Q} b_i(R, R') V^{2i}} =: G_V^{[P,Q]}(R, R') \quad (9)$$

in any bounded frequency range. Actually, it will be shown that a relatively low-order $(P = Q = 3)$ approximant does provide a quite reasonable overall accuracy. In the following we shall sketch a systematic procedure to find out an accurate low-order rational approximant of the spectral domain Green's function over the useful (single-mode) bandwidth. First, since the Green's function is analytical with respect to the frequency, one can expand it as a (truncated) McLaurin expansion[2]:

$$G_V(R, R') \approx \sum_{i=0}^{N} G_i(R, R') V^{2i} \qquad (10)$$

where the G_i functions can be computed by solving a hierarchy of *static* problems. In fact, taking into account the equivalent integral formulation of the Green's problem (7):

$$G_V(R, R') = G_0(R, R') - V^2 \cdot \int_0^1 R'' G_0(R, R'') G_V(R'', R') h(R'')\, dR'' \qquad (11)$$

where [12]

$$G_0(R, R') = \begin{cases} -I_0(WR)K_0(WR'), & \text{for } R < R' \\ -K_0(WR)I_0(WR'), & \text{for } R > R' \end{cases} \qquad (12)$$

$I_0(\cdot), K_0(\cdot)$ being modified Bessel function of zeroth order [15], and using (10) in (11) one readily finds the recursive rule [12]

$$\begin{aligned} G_i(R, R') &= -\int_0^1 R'' G_0(R, R'' G_{i-1}(R'', R') h(R'')\, dR'' \\ &= (-1)^i \int_0^1 \int_0^1 \cdots \int_0^1 R_i h(R_i) G_0(R, R_1) \\ &\quad \cdot G_0(R_i, R')\, dR_i \\ &\quad \cdot \prod_{j=1}^{i-1} R_j G_0(R_j, R_{j+1}) h(R_j)\, dR_j. \end{aligned} \qquad (13)$$

[2] Equation (10) can be recognized as a Stevenson series [14].

Starting from the (truncated) *analytic element* (10), one can obtain an effective rational function approximation for the spectral domain Green's function by exploiting the concept of analytic continuation and applying the Padé algorithm. This technique determines the coefficients a_i, b_i in (9) by requiring that the McLaurin expansion of the rational approximant (9) coincides up to the $(P+Q+1)$th term with (10). The rational function approximation (9) will be accordingly referred to as a (P, Q) Padé approximant. For conciseness, here we skip the detailed description of the Padé technique and its theoretical background; a thorough description may be found in [16] and [17]. Once the Padé approximant (9) of (8) is computed, the dominant eigenvalue and modal field can be readily deduced by extracting, respectively, the dominant (real, positive) pole and related residual, as seen from (8).

III. Numerical Implementation and Results

Let us summarize the required steps of the method.

- Evaluate the McLaurin expansion coefficients using (13) (which involves the computation of integrals up to $P+Q$ dimensions).
- Compute the rational approximant via Padé technique [which involves the sequential solution of two linear systems of size Q and P, yielding the a_i and b_i coefficients in (9)].
- Extract the dominant pole and related residual.

The first task is actually the most computationally demanding, requiring n-dimensional numerical quadratures $[1 \le n \le (P+Q)]$. However, in view of (12), one can conveniently split each one-dimensional integral in (13) so that the integrand function is *smooth* in each subintegral. Then, e.g., a gaussian quadrature [17] can be used to compute efficiently the $2n$ subintegrals.[3]

Once the coefficients are computed, evaluating the (P, Q) Padé approximant, its poles and residuals are relatively straightforward. Obviously, due to the unavoidable reduced-order modeling $(P, Q$ finite$)$ and the numerical approximations, the results (dispersion law and field distribution) will be affected by errors. Actually, a rigorous investigation on the effect of errors in the coefficients of the power series (10) on the accuracy of the Padé approximants is not yet available. However, a body of computational experiments suggest that, though the analytical continuation problem is indeed *ill-posed*, the Padé approximant (9) shows a surprising stability, as far as the estimation of the dominant pole is concerned [18], provided the order of the approximant is sufficiently high. In our numerical simulations we found that a (3, 3) Padé approximant provided a reasonable tradeoff between accuracy and computing times.

In order to illustrate the power of the proposed method we analyze two refractive index profiles for which analytical solutions are available, namely the *step* and the *quartic* profile [10]. For the latter we use as reference solution the power-series expansion (100 terms) reported in [10]. In Fig. 1 the relative accuracy of the approximate dispersion law in the

[3] In view of the mild behavior of the integrand functions, a three-points formula [17] is adequate.

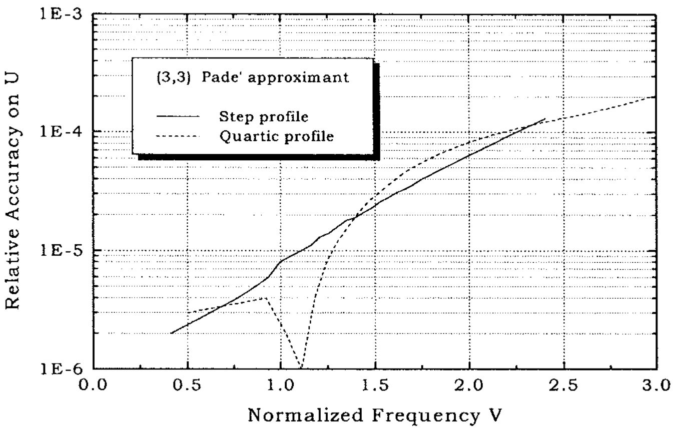

Fig. 1. (3, 3) Padé approximants. Relative accuracy on U for step and quartic profiles over the single-mode operation range.

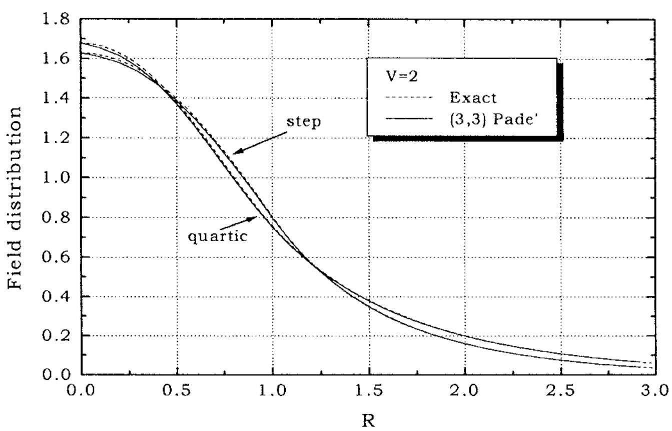

Fig. 2. Exact and Padé-computed field distributions for step and quartic profiles at $V = 2$.

single-mode frequency range, obtained using a diagonal $(P = Q = 3)$ Padé approximant, is reported for both profiles. As one can see, the accuracy is very good, especially in the low-frequency limit. Note that each point of the curve requires only 5 s to be computed on a PC Pentium P200. Fig. 2 displays the corresponding exact and approximate modal fields at a fixed frequency. Again, the agreement with the exact solution is excellent.

IV. CONCLUSIONS

We presented a novel effective numerical technique for analyzing single-mode fibers with arbitrary profile, based upon Padé approximation of the spectral domain Green's function. Computational features and comparison with known analytical results confirm that the proposed method allows to achieve uniformly high accuracy over the whole useful spectral range with very little memory and CPU time requirements. It is thus very well suited for fast and reliable cut and try optimization of the (transverse) refractive index profile.

REFERENCES

[1] J. P. Meunier, J. Pigeon, and J. N. Massot, "A general approach to the numerical determination of modal propagation constants and field distributions of optical fibers," *Opt. Quantum Electron.*, vol. 13, no. 1, pp. 71–83, 1981.

[2] S. M. Saad, "Review of numerical methods for the analysis of arbitrarily shaped microwave and optical dielectric waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 894–899, Oct. 1985.

[3] M. R. Shenoy, K. THyagarajan, and A. K. Gathak, "Numerical analysis of optical fibers using matrix approach," *J. Lightwave Technol.*, vol. 6, pp. 1285–1291, Aug. 1988.

[4] A. Kowalsky, "On the analysis of optical fibers described in terms of Chebyshev polynomials," *J. Lightwave Technol.*, vol. 8, pp. 164–167, Feb. 1990.

[5] R. L. Gallawa, I. C. Goyal, and A. K. Ghatak, "Modal properties of circular and noncircular optical waveguides," *Fiber Integ. Optics*, vol. 11, no. 1, pp. 25–50, Jan. 1992.

[6] V. Galdi, V. Pierro, and I. M. Pinto, "Cut-off frequency and dominant eigenfunction computation in complex dielectric geometries via Donsker-Kač formula and Monte Carlo method," *Electromagn.*, vol. 17, no. 1, pp. 1–14, 1997.

[7] P. Sircar and V. Arvind, "Numerical analysis of optical fibers with arbitrary profile," *J. Electromagn. Waves Applicat.*, vol. 11, pp. 1213–1227, 1997.

[8] S. J. Polychronopoulos, G. B. Athanasoulias, and N. K. Uzunoglu, "Advanced mode solver using an integral equation technique and entire domain plane wave basis functions," *Optical and Quantum Electronics*, vol. 29, no. 2, pp. 127–137, 1997.

[9] A. B. Manenkov and A. G. Rozhnev, "Optical dielectric waveguide analysis, based on the modified finite element and integral equation methods," *Opt. Quantum Electron.*, vol. 30, no. 1, pp. 61–70, 1998.

[10] A. W. Snyder and J. D. Love, *Optical Waveguide Theory*. London, U.K.: Chapman & Hall, 1983.

[11] S. C. Chao, W. H. Tsai, and M. S. Wu, "Extended gaussian approximation for single-mode fibers," *J. Lightwave Technol.*, vol. 10, pp. 22–27, Jan. 1992.

[12] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953.

[13] E. T. Titchmarsh, *The Theory of Functions*. Oxford, U.K.: Oxford Univ. Press, 1973.

[14] J. Van Bladel, *Electromagnetic Fields*. New York: Hemisphere, 1985.

[15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1964.

[16] J. S. R. Chisholm, "Mathematical theory of Padé approximants," in *Padé approximants*, P. R. Graves-Morris, Ed. London. U.K.: The Inst. of Physics, 1973.

[17] W. H. Press *et. al.*, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1992.

[18] J. L. Gammel, "Effect of random errors (noise) in the term of a power series on the convergence of the Padé approximants," in *Padé approximants*, P. R. Graves-Morris, Ed. London, U.K.: The Inst. of Physics, 1973.