ANALYTICAL APPROXIMATIONS FOR FUNDAMENTAL-MODE FIELD AND DISPERSION EQUATION OF PLANAR WAVEGUIDES THROUGH THE STEVENSON – PADÉ APPROACH

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ABSTRACT: *A novel method for the systematic derivation of analytical approximations of the fundamental-mode dispersion equation and field distribution of graded-index planar waveguides with uniform cladding is presented. The method is based on a Stevenson–Padé approach, recently introduced by the authors for the numerical analysis of graded-index optical fibers. A number of examples are presented in order to illustrate the power and accuracy of the method.* © 2000 John Wiley & Sons, Inc. Microwave Opt Technol Lett 27: 158–162, 2000.

Key words: *graded-index waveguides; Stevenson's series; Padé approximants*

I. INTRODUCTION

The modal analysis of graded-index planar waveguides has many important applications in the field of optical communications [1, 2]. In principle, modal analysis allows us to study guided propagation *exactly*. However, with a few exceptions (e.g., the infinite parabolic profile [2]), most profiles of practical interest do *not* admit analytical solutions in terms of simple functions, and even in the most common case, where one is interested in the properties of the fundamental mode only, one is led to numerically solve a dispersion equation involving special functions and/or series expansions [1, 2].

During the last decade, considerable effort has been devoted to developing approximate analytical methods to express the propagation constant and other quantities of interest in terms of simple analytical functions [1, 2] so as to provide both physical insight and readable guidelines for parameter optimization. Among these techniques, the Gaussian approximation [2], where the fundamental-mode field distribution of an arbitrary profile waveguide is approximated by a suitable Gaussian function, whose parameters are computed through a variational procedure, is probably the most popular. This approach yields elegant and compact approximations, but its accuracy is known to become poorer and poorer in the low-frequency limit [2].

Recently, we proposed an alternative framework [3, 4], based on the analytic continuation of Stevenson's expansion [5] via Padé approximants [6] for the efficient (fast and accurate) numerical analysis of single-mode optical fibers. The referred approach was shown to provide uniformly accurate results.

In this paper, we extend the above framework to planar waveguides, for which, in view of the simpler algebra involved, we are able to obtain uniformly accurate analytical approximations for the fundamental-mode dispersion law and field distribution over a reasonably wide spectral range.

The remainder of the paper is organized as follows. In Section II, the problem and the proposed approach are introduced. In Section III, a number of examples involving various profiles are presented. Conclusions follow under Section IV. A body of formal results is collected in the Appendixes.

II. BACKGROUND THEORY

A. Statement of the Problem. A symmetric graded-index planar waveguide with uniform cladding is considered. The refracting index profile is written as follows [2]:

$$n^2(x) = n_{co}^2[1 - 2\Delta f(x)], \qquad \Delta = \frac{n_{co}^2 - n_{cl}^2}{2n_{co}^2} \tag{1}$$

where n_{co} and n_{cl} are the maximum refraction index in the core and the (uniform) cladding index, respectively, and $f(x)$ is assumed to be an even function with $f(0) = 0$ and $f(x) = 1$ for $|x| \geq \rho$.

As is well known, these structures support transverse electric (henceforth TE) and transverse magnetic (henceforth TM) modes, and because of homogeneity and infinite extent along the y-direction, the fields do not depend on the y-variable. As usual, for guided fields, the z-dependence can be written as $\exp(\pm i\beta z)$, β being the propagation constant [2]. The x-dependence is ruled by the scalar Helmholtz equation [2]:

$$\left[\frac{d^2}{dX^2} + U^2 - V^2 f(X)\right]\phi^{(e)}(X) = 0 \qquad \text{(TE modes)} \tag{2}$$

$$\left[\frac{d^2}{dX^2} - \frac{1}{n^2(X)}\frac{dn^2(X)}{dX}\frac{d}{dX} + U^2 - V^2 f(X)\right]\phi^{(h)}(X) = 0 \qquad \text{(TM modes)} \tag{3}$$

where $X = x/\rho$, $\phi^{(e)}$, $\phi^{(h)}$ are suitable Debye potentials [2], and regularity-at-infinity boundary conditions are assumed. The dimensionless parameters V, U are defined as follows [2]:

$$V = k_0\rho(n_{co}^2 - n_{cl}^2)^{1/2}, \quad U = (V^2 - W^2)^{1/2}, \quad W = \rho(\beta^2 - k_0^2 n_{cl}^2)^{1/2}, \tag{4}$$

k_0 being the free-space wavenumber. An implicit $\exp(i\omega t)$ time-harmonic dependence is assumed throughout.

As is well known, at any fixed frequency, Eqs. (2), (3) admit a finite number of propagating (i.e., β real) modal solutions, the cutoff condition being defined by [2]

$$U = V, \quad \text{i.e.,} \quad W = 0. \tag{5}$$

In most applications, one is usually interested in the fundamental (minimum U at a fixed V) mode. It can be shown that, under the above assumptions, this latter is the lowest order even-symmetry TE solution (TE_0-mode) [2], with cutoff at $V = 0$ [2], and is accordingly the lowest order eigensolution of

$$\begin{cases} \left[\dfrac{d^2}{dX^2} - W^2 + V^2 h(X) \right] \phi^{(e)}(X) = 0, & X > 0 \\ \left. \dfrac{d\phi^{(e)}(X)}{dX} \right|_{X=0} = 0, & \lim_{X \to \infty} \phi^{(e)}(X) = 0. \end{cases} \tag{6}$$

Equations (6) follow from (2) after letting $h(X) = 1 - f(X)$ and placing a magnetic wall at $X = 0$ because of symmetry. The fundamental mode is completely described by its dispersion law, usually represented in the form $U = U_0(V)$, and field distribution $\phi_0^{(e)}(X)$.

B. Stevenson–Padé Approach. Let us briefly review the SP approach developed in [3, 4]. We start from an inhomogeneous version of (6):

$$\begin{cases} \left[\dfrac{d^2}{dX^2} - W^2 + V^2 h(X) \right] G_V(X) = g(X), & X > 0 \\ \left. \dfrac{dG_V(X)}{dX} \right|_{X=0} = 0, & \lim_{X \to \infty} G_V(X) = 0 \end{cases} \tag{7}$$

where $g(X)$ is an arbitrary (continuous) function whose choice will be discussed later. For any fixed positive value of W, Eqs. (6) admit a countable infinity of eigenvalues V_n^2 and corresponding eigenfunctions $\phi_n^{(e)}$ forming an orthonormal basis in $L^2(\mathscr{R})$ [7], so that $G_V(X)$ can be expressed *exactly* (L^2-convergence is understood) as [7]

$$G_V(X) = \sum_{n=0}^{\infty} \frac{\phi_n^{(e)}(X) \langle \phi_n^{(e)}, g \rangle}{V^2 - V_n^2} \tag{8}$$

where $\langle \cdot, \cdot \rangle$ denotes the weighted L^2 scalar product

$$\langle F, G \rangle := \int_0^{\infty} F(X) G(X) h(X)\, dX. \tag{9}$$

On the other hand, an alternative expression for G_V can be sought in the form of a Stevenson expansion [5], i.e., a McLaurin expansion in powers of V^2:

$$G_V(X) = \sum_{j=0}^{\infty} \hat{G}_j(X, W) V^{2j}. \tag{10}$$

Putting (10) in (7), and equating to zero all coefficients[1] of the resulting series in powers of V^2, one readily obtains a *hierarchical* system of *static* ($V = 0$) problems [3, 4] in the unknown kernels $\hat{G}_j(X, W)$, viz.

$$\begin{cases} \left[\dfrac{d^2}{dX^2} - W^2 \right] \hat{G}_j(X, W) = H_j(X, W), & X > 0, j \ge 0 \\ H_0(X, W) = g(X) \\ H_j(X, W) = -h(X) \hat{G}_{j-1}(X, W), & j > 0 \end{cases} \tag{11}$$

[1] This is equivalent to repeatedly differentiating the resulting identity with respect to V^2 and setting $V = 0$.

where the same boundary conditions as in (7) are imposed for all j. The solutions of the above system are formally obtained [7] by convolving the functions H_j on the right-hand side of (11) with the static Green's function, viz.

$$\hat{G}_j(X, W) = \int_0^{\infty} \tilde{G}(X, X'; W) H_j(X', W)\, dX', \qquad j \ge 0 \tag{12}$$

$$\tilde{G}(X, X'; W) = -\frac{1}{2W} \{ \exp(-W|X - X'|) + \exp[-W(X + X')] \}. \tag{13}$$

The SP approach developed in [3, 4] is based on the analytical continuation of (10) into a rational approximant. This latter can be compared to the (exact) meromorphic representation (8) to estimate the sought eigenvalues (poles) V_n^2 and eigenfunctions $\phi_n^{(e)}$.

A systematic procedure to construct rational approximants starting from a power series is provided by the Padé algorithm [6]. Due to space limitations, here, we skip all mathematical properties and technical issues, for which the interested reader might refer to [6]. It will suffice here to note that the (P, Q) Padé approximant (henceforth PA) of G_V is the ratio

$$G_V^{(P,Q)} = \frac{\sum_{p=1}^{P} a_p V^{2p}}{1 + \sum_{q=1}^{Q} b_q V^{2q}} \tag{14}$$

where the unknown coefficients $a_p (p = 1, 2, \dots, P)$, $b_q (q = 1, 2, \dots, Q)$ can be retrieved by solving a linear system involving the first $P + Q + 1$ coefficients of the original power series, obtained by enforcing equality between this latter and the McLaurin expansion of (14) to order $P + Q$ [6].

The above-sketched procedure was successfully applied in [3, 4] to (circular) optical fibers. It was shown that a (3,3) PA was able to provide remarkable accuracy (five decimal figures, on average) for both the fundamental-mode distribution and the related dispersion law over the whole single-mode range. In [3, 4], the required Stevenson kernels were computed numerically, using standard multidimensional quadrature algorithms [3, 4].

For planar waveguides, thanks to the simpler form of (13), one can demonstrate by complete induction that, whenever $f(X)$ in (1) and $g(X)$ in (7) are superpositions of simple terms of the form

$$\exp(aX) X^k, \qquad a \in C, k \in \mathscr{N}, \tag{15}$$

the integrals (12) can be computed in closed forms, the resulting functions H_j being in turn superpositions of functions of the same type (15). As a result, for planar waveguides, one can derive, in principle, *analytic* approximations of arbitrary high-order modes and dispersion equations.[2]

In particular, low-order PAs yield quite efficient *analytical* approximations of the fundamental-mode field and dispersion law. In the following, we will use the simplest (1,1) PA to

[2] Note that, in principle, aside from the rapidly increasing size of the high-order terms, analytic solutions are not affected by ill conditioning, which limits the applicability of the numerical approach developed in [3, 4] to higher order modes.

approximate G_V as

$$G_V^{(1,1)}(X) = \frac{a_0(X,W) + a_1(X,W)V^2}{1 + b_1(X,W)V^2}. \tag{16}$$

The Padé coefficients a_0, a_1, b_1 are easily found to be [6]

$$a_0(X,W) = \hat{G}_0(X,W) \tag{17}$$

$$a_1(X,W) = \hat{G}_1(X,W) - \hat{G}_0(X,W)\frac{\hat{G}_2(X,W)}{\hat{G}_1(X,W)} \tag{18}$$

$$b_1(X,W) = -\frac{\hat{G}_2(X,W)}{\hat{G}_1(X,W)}. \tag{19}$$

Identifying the (only) pole and related residual of (16) with the $n = 0$ pole and residual of (8) allows us to easily read off V_0 and $\phi_0^{(e)}$, viz.

$$V_0 \approx \left[-\frac{1}{b_1(X,W)}\right]^{1/2} = \left[\frac{\hat{G}_1(X,W)}{\hat{G}_2(X,W)}\right]^{1/2} \tag{20}$$

$$\phi_0^{(e)}(X) \approx -\frac{\hat{G}_1^3(X,W)}{\hat{G}_2^2(X,W)}. \tag{21}$$

The dispersion equation (20) is (weakly!) dependent on X as an effect of the involved approximation. The dispersion curves obtained from (20) for any $X \in (0,1)$ are almost indistinguishable. The special choice $X = 0$ yields the simplest result, and will be used in the sequel.

A few comments on the implementation of the method are in order. Equation (20) gives the dispersion law in the unusual form $V = V(W)$ which, together with (4), yields the standard form $U = U(V)$ in implicit form. We note that the proposed approximation, unlike the Gaussian one, performs very well in the low-frequency limit, being based on the McLaurin expansion in powers of the (squared) frequency.

As far as the choice of the function g in (7) is concerned, a judicious choice of g can considerably improve the accuracy of the method. In fact, it is readily recognized from (8) that the special choice $g = \phi_0^{(e)}$ makes G_V a single-pole function due to the eigenfunction orthogonality, so that the (1,1) PA recovers the *exact* solution [6]. Obviously, such a choice is impossible, $\phi_0^{(e)}$ being unknown *a priori*. It can be understood that g should be chosen as close as possible to $\phi_0^{(e)}$. Furthermore, the choice of g should also take into account the possibility of simplifying the analytical evaluation of the Stevenson kernels (12). We used

$$g(X) = 2W\exp(-WX) \tag{22}$$

which provides a reasonable tradeoff.

The simple, but lengthy analytical evaluation of the needed Stevenson kernels $\hat{G}_0$, $\hat{G}_1$, $\hat{G}_2$ can be systematically performed using computer-aided algebraic manipulation tools, such as, e.g., Mathematica™ [8].

III. SOME RESULTS

As an illustration of our method, in this section, we present and discuss some results obtained applying the proposed SP approach to various profiles of the type (15). All relevant algebraic manipulations were performed using Mathematica™ [8].

A. Step Profile. We consider first the simplest example of a step profile, for which an exact solution is available:

$$f(X) = \begin{cases} 0, & 0 \le X < 1 \\ 1, & X \ge 1. \end{cases} \tag{23}$$

The Stevenson kernels take the simplest form:

$$\hat{G}_0(X,W) = -\left(\frac{1}{W} + X\right)\exp(-WX)$$

$$\hat{G}_j(X,W) = \begin{cases} P_j(X,W)\exp(-WX) + Q_j(X,W) \\ \quad \times\exp(WX), \qquad 0 \le X < 1 \\ [P_j(1,W) + Q_j(1,W)\exp(2W)] \\ \quad \times\exp(-WX), \qquad X \ge 1 \end{cases} \qquad j > 0 \tag{24}$$

where P_j and Q_j are polynomials in X of degree $j+1$ and $j-1$, respectively, collected in Appendix A. Using (20) and (21) together with (24), the SP fundamental-mode field distribution and dispersion law are readily obtained.

The SP field distribution (whose detailed expression is omitted for brevity) is compared in Figure 1(a) to the known exact solutions for three different values of W close to the

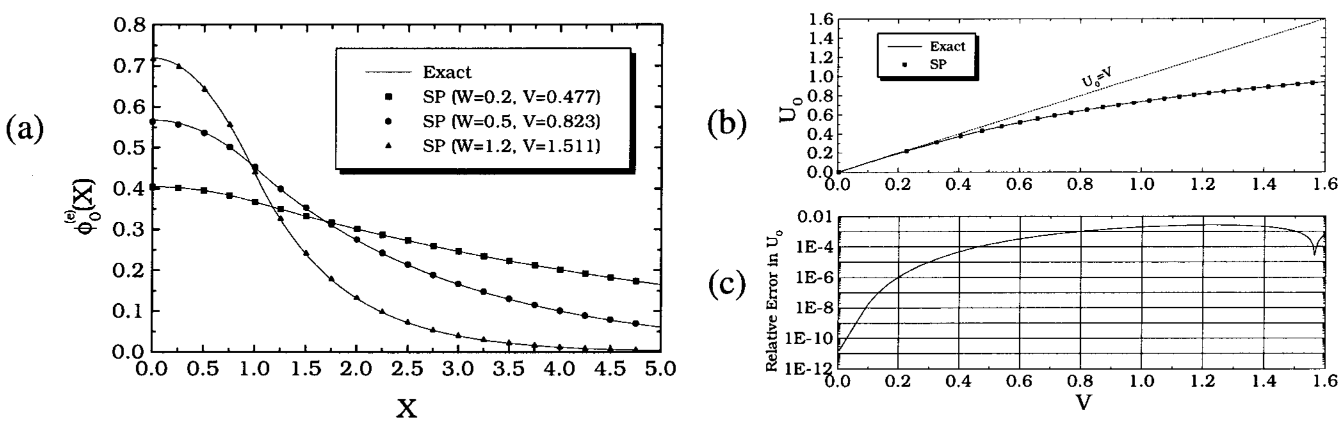

Figure 1 Step profile. (a) Exact and SP fundamental-mode field distributions at various frequencies. (b) Exact and SP fundamental-mode dispersion laws. (c) SP dispersion-law relative error

edges and center of the dominant mode spectral range. The agreement is excellent.

The SP fundamental-mode dispersion law has the simple form

$$V_0(W) \approx \left[\frac{\hat{G}_1(0,W)}{\hat{G}_2(0,W)}\right]^{1/2} = 2W\alpha\left[\frac{3-3\alpha^2+2W}{(13+16W+6W^2)\alpha^2-10\alpha^4-2W-3}\right]^{1/2}, \quad \alpha = \exp(W) \quad (25)$$

which, together with (4), yields the $U = U(V)$ curve shown in Figure 1(b). The excellent agreement with the exact result throughout the dominant mode spectral range is better quantified in terms of the relative error, shown in Figure 1(c).

Note that Eq. (25) exhibits a $0/0$ indeterminate form at cutoff ($W = 0$). The cutoff ($W \to 0$) limit is accurately described by the following simple expansion of (25) in fractional powers of W:

$$V_0(W) \approx W^{1/2} + \frac{W^{3/2}}{3} - \frac{W^{5/2}}{48} + O(W^{7/2}). \quad (26)$$

The last equation turns out to be considerably more accurate than, e.g., the one reported in [2, sect. 12.4].

B. Quadratic Profile. As a next nontrivial example, we consider the family

$$f(X) = \begin{cases} aX^2 + (1-a)X, & 0 \le X < 1, \quad 0 \le a \le 1 \\ 1, & X \ge 1 \end{cases} \quad (27)$$

which describes a variety of quadratic profiles, as shown in Figure 2. We omit for brevity the explicit expression of the Stevenson's kernels $\hat{G}_j$, and focus on the SP fundamental-mode dispersion law, which is the main design figure of interest, viz.

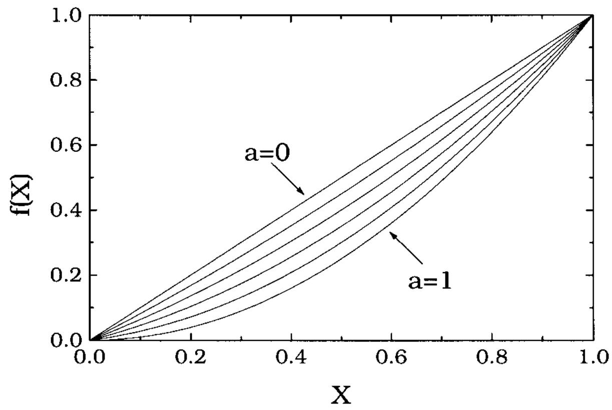

Figure 2 Quadratic profiles [Eq. (27)]

$$V_0(W) \approx 2\sqrt{6}\,\alpha W^2 \times \left[\frac{F_1(W,a) + \alpha^2 F_2(W,a)}{F_3(W,a) + \alpha^2 F_4(W,a) + \alpha^4 F_5(W,a)}\right]^{1/2}, \quad \alpha = \exp(W) \quad (28)$$

where $F_1, \ldots, F_5$ are polynomials in W, a, collected in Appendix B. Equation (28) is relatively handy, and is compared to a reference power-series solution ([2], truncated so as to guarantee 15 accurate decimal figures) in Figure 3(a) for some representative values of a throughout the dominant mode spectral range. The agreement is again excellent, as can be seen from the pertinent errors shown in Figure 3(b).

A simple and accurate (fractional) power-series approximation of (28) valid nearby cutoff ($W \approx 0$) can be obtained, similar to (26), viz.

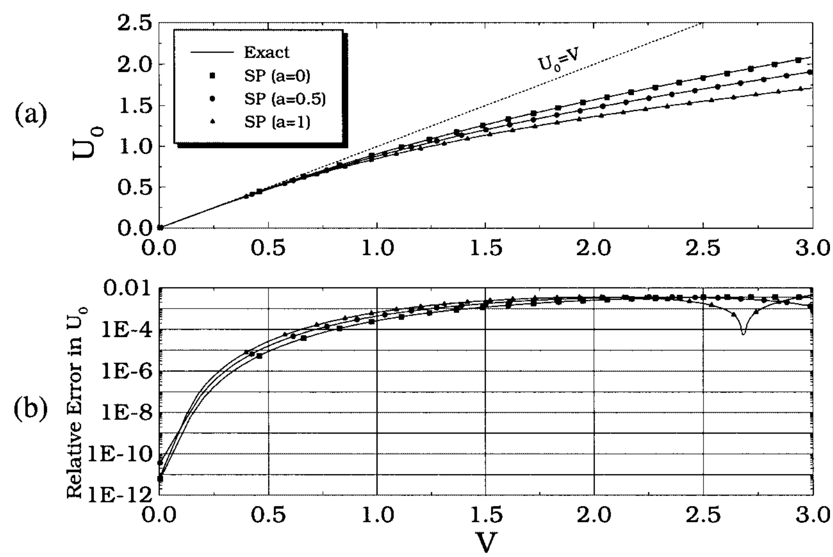

Figure 3 Quadratic profiles [Eq. (27)]. (a) Reference (power series [2]) and SP fundamental-mode dispersion laws at various values of a. (b) SP dispersion-law relative error

$$V_0(W) \approx \sqrt{\frac{6}{3+a}}\, W^{1/2} + \sqrt{\frac{3}{2(3+a)}}\left[\frac{147+119a+22a^2}{35(3+a)^2}\right]W^{3/2} - \sqrt{\frac{3}{2(3+a)}} \times \left[\frac{29106+38724a+17171a^2+3374a^3+281a^4}{9800(3+a)^4}\right]W^{5/2} + O(W^{7/2}). \quad (29)$$

IV. CONCLUSIONS

A novel approach based upon the combined use of Stevenson series and Padé approximant-based analytic continuation has been proposed for the systematic derivation of efficient (accurate and relatively simple) analytical approximations of the fundamental-mode dispersion law and field distribution in planar graded-index dielectric waveguides with a symmetric profile. The involved calculations can be easily performed using off-the-shelf algebraic-manipulation software tools.

As a check, the proposed approach has been applied to a number of different profiles, yielding quite accurate results, over the dominant mode spectral range. In particular, simple and accurate dispersion law representations valid near cutoff have been derived.

The method can be extended, in principle, to odd modes (and/or grounded slabs) by simply changing the + sign between the exponentials on the right-hand side of (13). However, PAs of order higher than (1,1) might be needed to achieve comparable accuracy, in view of the higher frequencies involved.

APPENDIX A

$$P_1(X,W) = \left(\frac{3-6\alpha^2+2W}{8\alpha^2W^3}\right) - \frac{3X}{4W^2} - \frac{X^2}{4W}, \quad \alpha = \exp(W) \quad (A1)$$

$$Q_1(X,W) = \frac{3+2W}{8\alpha^2W^3} \quad (A2)$$

$$P_2(X,W) = \left[\frac{-3+(13+16W+6W^2)\alpha^2-20\alpha^4-2W}{32\alpha^4W^5}\right] + \left(\frac{3-10\alpha^2+2W}{16\alpha^2W^4}\right)X - \frac{X^2}{4W^3} - \frac{X^3}{24W^2} \quad (A3)$$

$$Q_2(X,W) = \left[\frac{-3+(13+16W+6W^2)\alpha^2-2W}{32\alpha^4W^5}\right] - \left(\frac{3+2W}{16\alpha^2W^4}\right)X. \quad (A4)$$

APPENDIX B

$$F_1(W,a) = 5a + (4+6a)W + 2(a+1)W^2 \quad (B1)$$

$$F_2(W,a) = -5a + 4(a-1)W + 6W^2 \quad (B2)$$

$$F_3(W,a) = 15a^2 + (27+33a)aW + (12+36a+24a^2)W^2 + (6+12a+6a^2)W^3 \quad (B3)$$

$$F_4(W,a) = -465a^2 - (501a+429)aW - (120+276a+192a^2)W^2 + (24-16a)aW^3 + (48+76a+18a^2)W^4 + (16+22a+6a^2)W^5 \quad (B4)$$

$$F_5(W,a) = 450a^2 + 402(1-a)aW + (108-510a+108a^2)W^2 + 198(a-1)W^3 + 120W^4. \quad (B5)$$

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APPLICATION OF ON-SURFACE MEI METHOD IN ANALYSIS OF TRANSMISSION LINES

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ABSTRACT: *In this letter, the on-surface measured equation of invariance (OSMEI) is applied to a class of electrostatic closed-region problem, i.e., transmission lines composed of inner and outer conductors. Since it results in a highly sparse final matrix, and the computational domain is limited to the surface of the inner conductor, the algorithm is very memory economic. Two typical transmission lines are analyzed, and the results show excellent agreement with those of other methods.* © 2000 John Wiley & Sons, Inc. Microwave Opt Technol Lett 27: 162–165, 2000.

Key words: *on-surface measured equation of invariance; transmission line; characteristic impedance*