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Abstract—We propose a new, fast, and accurate numerical technique for analyzing single-mode optical fibers with arbitrary (transverse) refractive index profile. The method is based upon a Padé (rational) approximation of the spectral domain Green’s function of the fiber, obtained by solving a hierarchy of static problems. The sought eigenfrequency and modal field are accordingly estimated by computing, respectively, the dominant pole and the related residual of the rational approximant. Numerical simulations and comparison with known analytical results indicate that the proposed method is highly accurate, reliable, and computationally affordable.

Index Terms—Optical fibers, Padé approximants.

I. INTRODUCTION

The ever-increasing interest in numerical modeling of optical fibers is witnessed by the huge amount of recent technical papers devoted to this subject [1]–[9]. Most of the fibers currently manufactured for long-distance communication systems are weakly-guiding and operate in the single-mode range. In this connection, the design of the profile index function is usually obtained by trial and error analysis so as to optimize the dispersion characteristics and matching them to the spectral features of specific sources. A practical implementation of this procedure requires a fast and reliable numerical tool for solving the scalar wave equation. As a matter of fact, analytical solutions are available only for few profiles; standard analytical approximation techniques (e.g., gaussian [10], [11]) cannot be applied for arbitrary profiles and are known to fail in the low-frequency limit, while available numerical techniques (including matrix approaches [3], polynomial expansions [4], [7], Galerkin methods [1], [5], [8], finite elements [9], and path-integrals [6]) become inefficient in terms of memory allocation and/or computing times if highly accurate results are required.

In this letter we present a novel approach, based upon a Padé approximation of the spectral domain Green’s function, which is shown to provide a very accurate approximation for the dispersion law and the field distribution of the fundamental mode over the useful spectral range, for any (transverse) index distribution, with minimum storage requirements and CPU times.

II. THE METHOD

A longitudinally invariant graded-index circular fiber with uniform cladding is considered. An \( \exp(-j\omega t) \) time-harmonic dependence is implicitly assumed throughout the remainder of the paper. The refractive index profile is expressed as [10]

\[
n^2(r) = n_{co}^2[1 - 2\Delta f(r)], \quad f(r) = \begin{cases} 0, & \text{for } r = 0 \\ 1, & \text{for } r \geq \rho \end{cases}
\]

where \( r \) is the radial coordinate and \( \Delta = (1 - n_{cl}^2/n_{co}^2)/2 \) is the so-called profile height parameter, \( n_{co}, n_{cl} \) being the maximum core index and the clad index, respectively.

where \( k \) is the free-space wavenumber and \( \beta \) is the propagation constant.

In the weakly-guiding regime (\( \Delta \ll 1 \)), the radial behavior of the HE\(_{2n}\) modes is ruled by the following Sturm–Liouville equation [10]:

\[
\left[ \frac{1}{R} \frac{d}{dR} \left( R \frac{d}{dR} \right) - W^2 + V^2 h(R) \right] \phi(R) = 0
\]

where \( R = r/\rho \) and \( h(R) = 1 - f(R) \).

For any fixed positive value of \( W^2 \) (guided modes) the above problem admits a discrete infinity of eigenvalues \( \{V_n^2\} \) and related eigenfunctions \( \{\phi_n\}, n = 1, 2, 3, \ldots \), which form an orthonormal basis for \( L^2(\mathbb{R}^2) \), i.e., [12]

\[
\int_0^1 R \phi_n(R) \phi_m(R) h(R) dR = \delta_{nm}
\]

\( \delta_{nm} \) being the Kronecker symbol. Our aim is to find out the dispersion relation (usually represented as a \( U \) versus \( V \) diagram) and the fundamental mode (HE\(_{11}\)). To this end, let

1 Extension to the more general vector (or coupled-scalar) problem, though possible in principle, entails considerable formal complications. In this letter we limit ourselves to the simpler (though highly meaningful) case of weakly guiding fibers to highlight the physical content of the method.
us consider the spectral domain Green’s function (ring source) [12]:

\[
\left[ \frac{1}{R} \frac{d}{dR} \left( R \frac{d}{dR} \right) - W^2 + V^2 h(R) \right] G_V(R, R') = \frac{1}{R} \delta(R - R')
\]

which admits an expression in terms of the aforementioned eigenvalues and eigenfunctions, viz., [12]

\[
G_V(R, R') = \sum_{n=1}^{\infty} \frac{\phi_n(R) \phi_n(R')}{V^2 - \nu_n^2}.
\]

Since \( G_V \) is meromorphic, Mittag-Leffler’s theorem [13] suggests that it could be well approximated by a rational function of the (square of the) frequency, viz.,

\[
G_V(R, R') \approx \sum_{i=0}^{P} \frac{a_i(R, R')}{V^{2i}} + \sum_{i=1}^{Q} \frac{b_i(R, R')}{V^{2i}} = G_V^{P,Q}(R, R')
\]

in any bounded frequency range. Actually, it will be shown that a relatively low-order \((P = Q = 3)\) approximant does provide a quite reasonable overall accuracy. In the following we shall sketch a systematic procedure to find out an accurate low-order rational approximant of the spectral domain Green’s function over the useful (single-mode) bandwidth. First, since the Green’s function is analytical with respect to the frequency, one can expand it as a (truncated) McLaurin expansion \(2\):

\[
G_V(R, R') \approx \sum_{i=0}^{N} G_i(R, R') V^{2i}
\]

where the \( G_i \) functions can be computed by solving a hierarchy of static problems. In fact, taking into account the equivalent integral formulation of the Green’s problem (7):

\[
G_V(R, R') = G_0(R, R') - V^2 \int_0^1 R'^i G_0(R, R') G_0(R'^{2i}, R') h(R') \, dR'
\]

where [12]

\[
G_0(R, R') = \begin{cases} -I_0(WR)K_0(WR'), & \text{for } R < R' \\ -K_0(WR)I_0(WR'), & \text{for } R > R' \end{cases}
\]

\( I_0(\cdot), K_0(\cdot) \) being modified Bessel function of zeroth order [15], and using (10) in (11) one readily finds the recursive rule [12]

\[
G_i(R, R') = (-1)^i \int_0^1 \int_0^1 \cdots \int_0^1 R'^i G_0(R, R'^{2i}) h(R') \, dR' \\
\cdot \prod_{j=1}^{i-1} R_j G_0(R_j, R_{j+1}) h(R_j) \, dR_j.
\]

III. NUMERICAL IMPLEMENTATION AND RESULTS

Let us summarize the required steps of the method.

- Evaluate the McLaurin expansion coefficients using (13) (which involves the computation of integrals up to \( P + Q \) dimensions).
- Compute the rational approximant via Padé technique [which involves the sequential solution of two linear systems of size \( P \) and \( Q \), yielding the \( a_i \) and \( b_i \) coefficients in (9)].
- Extract the dominant pole and related residual.

The first task is actually the most computationally demanding, requiring \( n \)-dimensional numerical quadratures \([1 \leq n \leq (P + Q)]\). However, in view of (12), one can conveniently split each one-dimensional integral in (13) so that the integrand function is smooth in each subintegral. Then, e.g., a gaussian quadrature [17] can be used to compute efficiently the \( 2n \) subintegrals.\(^3\)

Once the coefficients are computed, evaluating the \((P, Q)\) Padé approximant, its poles and residuals are relatively straightforward. Obviously, due to the unavoidable reduced-order modeling \((P, Q \text{ finite})\) and the numerical approximations, the results (dispersion law and field distribution) will be affected by errors. Actually, a rigorous investigation on the effect of errors in the coefficients of the power series (10) on the accuracy of the Padé approximants is not yet available. However, a body of computational experiments suggest that, though the analytical continuation problem is indeed ill-posed, the Padé approximant (9) shows a surprising stability, as far as the estimation of the dominant pole is concerned [18], provided the order of the approximant is sufficiently high. In our numerical simulations we found that a \((3, 3)\) Padé approximant provided a reasonable tradeoff between accuracy and computing times.

In order to illustrate the power of the proposed method we analyze two refractive index profiles for which analytical solutions are available, namely the step and the quartic profile [10]. For the latter we use as reference solution the power-series expansion (100 terms) reported in [10]. In Fig. 1 the relative accuracy of the approximate dispersion law in the

\(^3\)In view of the mild behavior of the integrand functions, a three-points formula [17] is adequate.
single-mode frequency range, obtained using a diagonal \((P = Q = 3)\) Padé approximant, is reported for both profiles. As one can see, the accuracy is very good, especially in the low-frequency limit. Note that each point of the curve requires only 5 s to be computed on a PC Pentium P200. Fig. 2 displays the corresponding exact and approximate modal fields at a fixed frequency. Again, the agreement with the exact solution is excellent.

**IV. CONCLUSIONS**

We presented a novel effective numerical technique for analyzing single-mode fibers with arbitrary profile, based upon Padé approximation of the spectral domain Green’s function. Computational features and comparison with known analytical results confirm that the proposed method allows to achieve uniformly high accuracy over the whole useful spectral range with very little memory and CPU time requirements. It is thus very well suited for fast and reliable cut and try optimization of the (transverse) refractive index profile.

**REFERENCES**


